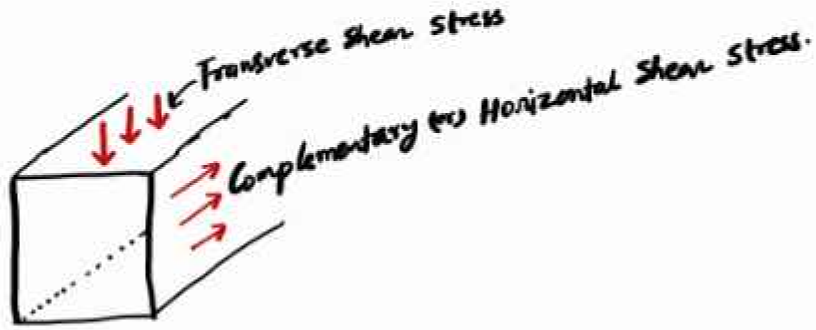
Shear Stresses

- Shear Stresses
- Derivation of shear stress formula
- Shear Stress distribution across various beam sections

Previous chapter was discussed about variation of bending stresses from point to point in a beam by ignoring the effect of shearing forces.



The applied shearing force will be distributed as a shearing stress across transverse sections of the beam. At the each point on the section the transverse shearing stress will produce a complimentary 'horizontal' shearing stress.

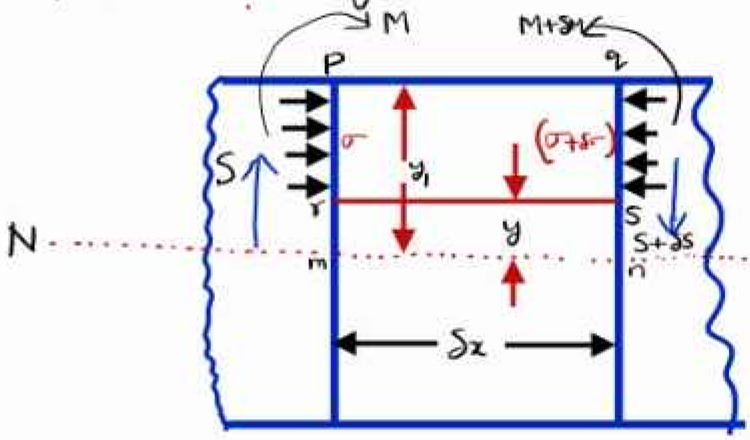
Shearing stresses acting between sucessive longitudinal layer of the beam, tending to resist sliding between these layers.

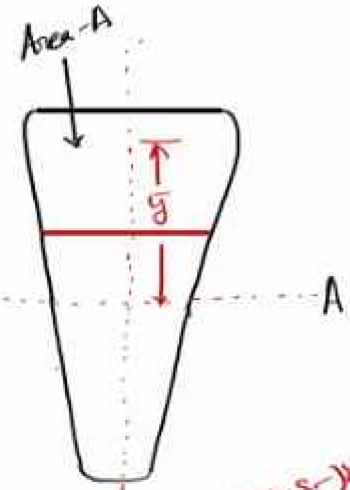
The longitudinal shearing stresses will balance the variation of bending stresses along the beam.

Concept of Shearing Stress variation

Consider a small slice of a beam of length δx, with a variation of bending moment over its length from M to $(M+\delta M)$. Consider a layer rs at a distance y from neutral axis. Let the width of the layer be b and hence its area $(\delta x \times b)$.

$S, S+\delta S$ = Shear forces at two sections
$M, M+\delta M$ = Bending moments at the two sections

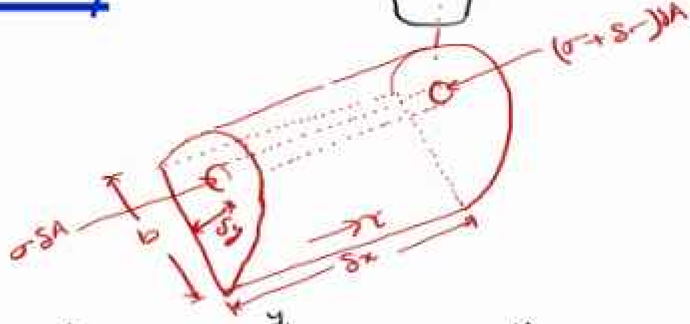




From Bending equation

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\sigma = \frac{M}{I} \cdot y$$



The total force on the face pr $= \sum_{y}^{y_1} \sigma \cdot A = \int_{y}^{y_1} \frac{M}{I} \cdot (\delta y \times b) = \int_{y}^{y_1} \frac{M}{I} (yb)\, dy$

and total force on sq $= \int_{y}^{y_1} \frac{(M+\delta M)}{I} (y \cdot b)\, dy$

The excess force between sq and rp

$$= \int_{y}^{y_1} \frac{\delta M}{I} (y \times b)\, dy = \frac{\delta M}{I} \int_{y}^{y_1} (yb)\, dy$$

But, $\int_{y}^{y_1} y \cdot b\, dy$ is moment of area of the face pr about the neutral axis

(or) $\int_{y}^{y_1} y \cdot b\, dy = A\bar{y}$

The excess force will be balanced by a Shearing Stress τ acting along rs. The force due to this stress is $\tau \times (b \cdot \delta x)$

$$\therefore \tau \times (b \cdot \delta x) = \frac{\delta M}{I} \cdot A\bar{y} \quad \text{or} \quad \tau = \frac{\delta M}{\delta x} \times \frac{1}{Ib} \cdot A\bar{y}$$

$\frac{\delta M}{\delta x} = S$, the Shearing Stress $\boxed{\tau = \frac{SA\bar{y}}{Ib}}$

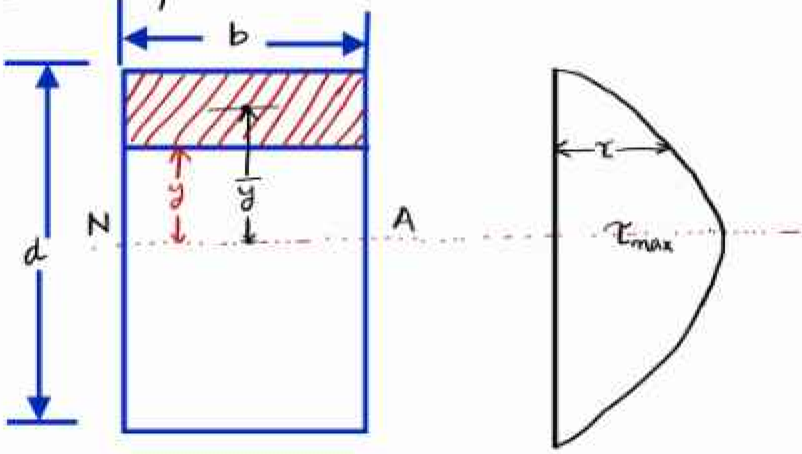
Shearing force.

Where, b is the width of the beam at the point where the Shearing force is considered, and $A\bar{y}$ is the moment of sectional area above that point about the neutral axis.

Variation in shear Stress for beam cross-section

(1) Rectangular Section

Consider b & d are the width and depth of a beam cross-section. τ be the shear Stress in a layer at a distance y from N.A. where the section is subjected to Shear force.



$$\bar{y} = y + \frac{\left(\frac{d}{2} - y\right)}{2}$$
$$= y + \frac{d}{4} - \frac{y}{2}$$
$$= \frac{y}{2} + \frac{d}{4}$$
$$\boxed{\bar{y} = \frac{1}{2}\left(y + \frac{d}{2}\right)}$$

Area of shaded part $= A = b\left(\frac{d}{2} - y\right)$, $\bar{y} = \frac{1}{2}\left(y + \frac{d}{2}\right)$

$$I = \frac{bd^3}{12}$$

$$A\bar{y} = b\left(\frac{d}{2} - y\right) \times \frac{1}{2}\left(y + \frac{d}{2}\right) = \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

$$\tau = \frac{SA\bar{y}}{I \cdot b} = \frac{S \times \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)}{\frac{bd^3}{12} \times b}$$

$$\tau = \frac{12S \times b}{b \times bd^3 \times 2}\left(\frac{d^2}{4} - y^2\right)$$

This section will have maximum shear stress (τ) at the N.A. because y value will be zero.

$$(\text{or}) \quad \tau_{max} = \frac{12S}{bd^3} \times \frac{d^2}{8} = \frac{3}{2} \times \frac{S}{bd}$$

Here, $\frac{S}{bd}$ is mean shear stress

So, $\tau_{mean} = \frac{S}{bd}$

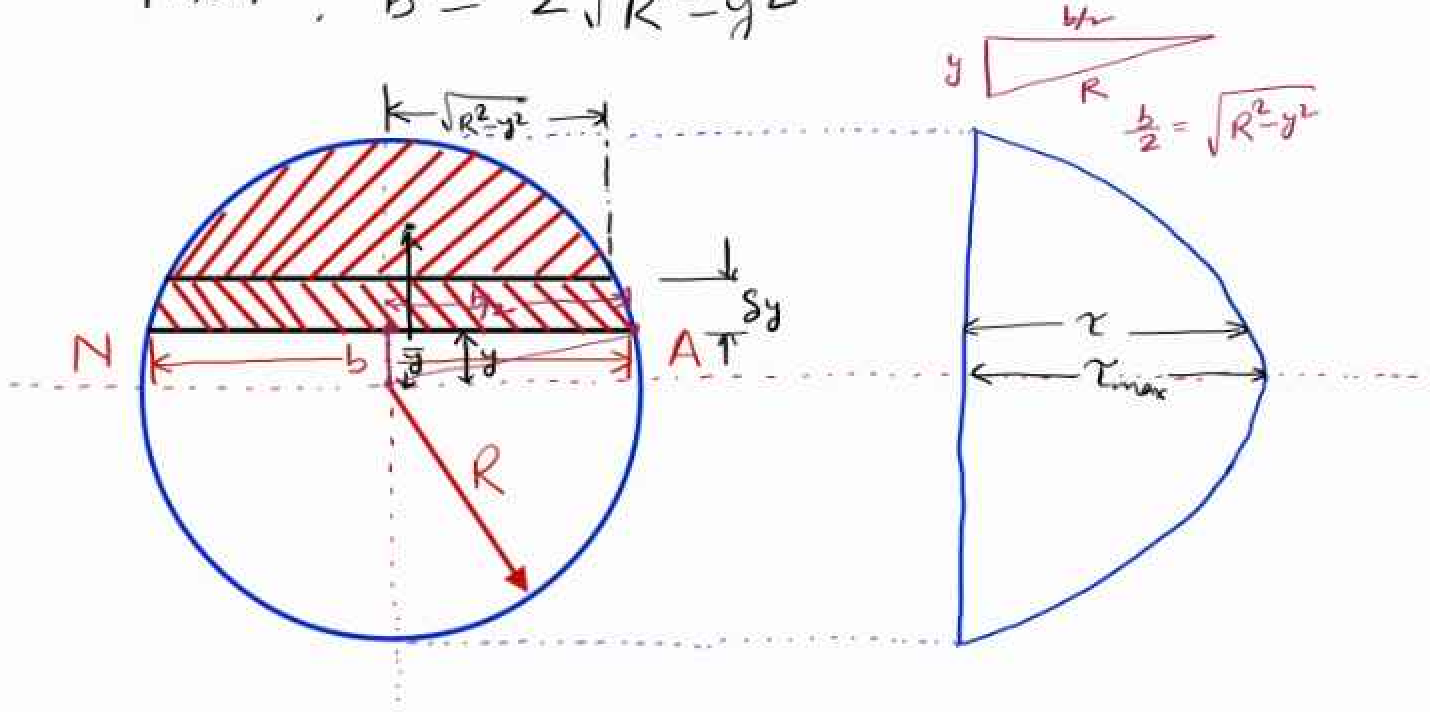
$$\boxed{\tau_{max} = \frac{3}{2}\tau_{mean}}$$

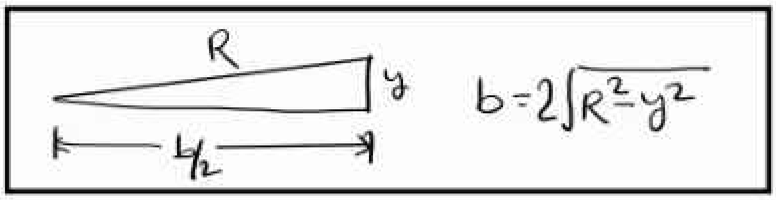
(2) Circular Section

Consider a solid circular section of radius R.

Take a elementary strip of thickness δy at a distance y from N.A. and width of strip be 'b'.

Then, $b = 2\sqrt{R^2-y^2}$





Area of the elementary strip, $\delta a = b \times \delta y$

$$= 2\sqrt{R^2-y^2} \times \delta y$$

Moment of elemental area about neutral axis $= \delta a \times y$

$$= 2\sqrt{R^2-y^2} \cdot y \cdot \delta y$$

$$= \int_y^R 2y\sqrt{R^2-y^2}\, dy = \int_y^R b \cdot y\, dy$$

$\because \; b = 2\sqrt{R^2-y^2} \qquad \therefore \; b^2 = 4(R^2-y^2)$

differentiating both sides

$$2b \times bd = -4 \times 2y\,dy$$

(or) $y\,dy = -\dfrac{b\cdot db}{4}$

when, at $y = y$, $b = b$

at $y = R$, $b = 0$

$$\therefore \int_y^R b\cdot y\cdot dy = \int_b^0 -b\cdot\frac{b\,db}{4} = \int_b^0 -\frac{b^2\,db}{4} = \left[\frac{b^3}{12}\right]_0^b = \frac{b^3}{12}$$

Now, $\tau = \dfrac{S}{Ib} \times \underline{\text{moment of shaded area } (A\bar{y})}$

$$\tau = \frac{S}{Ib} \times \frac{b^3}{12}$$

$$= \frac{S}{12I}b^2 = \frac{S}{12 \times \frac{\pi R^4}{4}}\left[4(R^2 - y^2)\right]$$

$$\tau = \frac{4}{3}\frac{S}{\pi R^4}(R^2 - y^2)$$

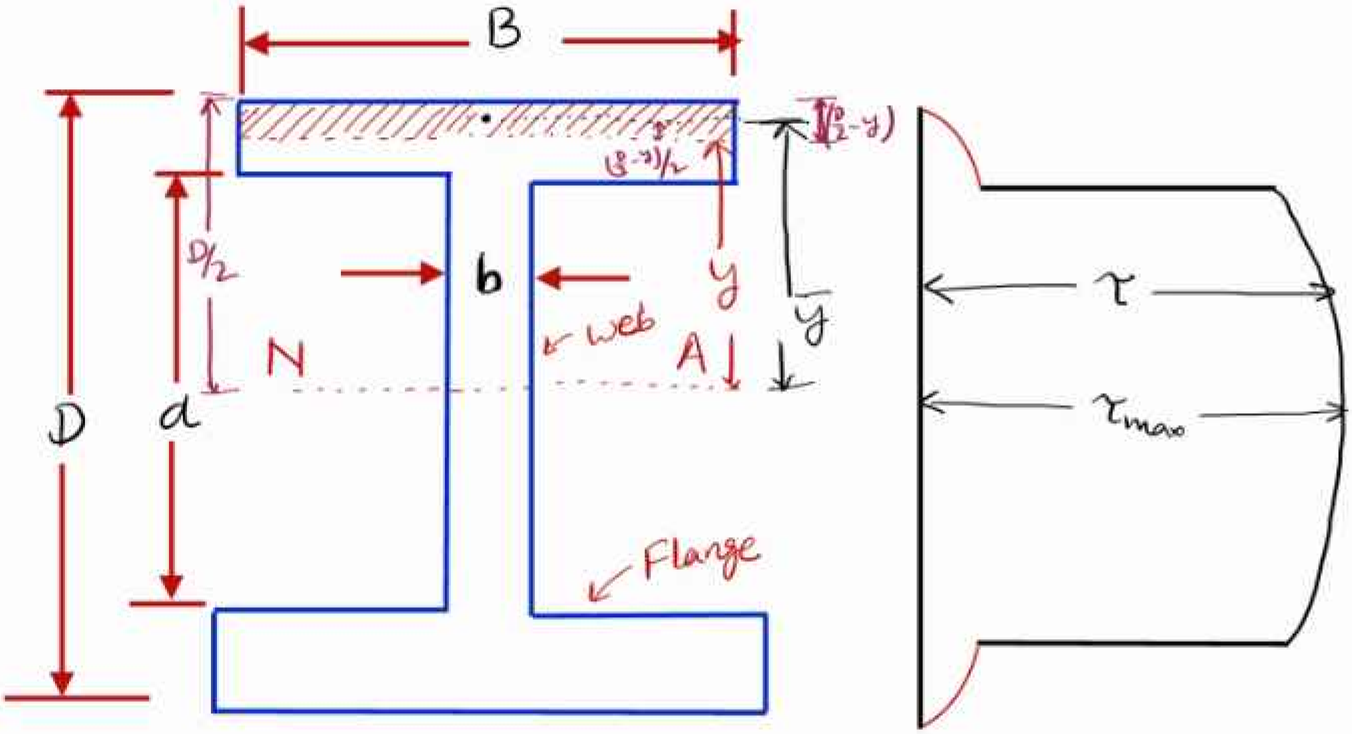
$$\tau = \frac{4}{3}\frac{S}{\pi R^4}(R^2 - y^2)$$

at $y = R$, $\tau = 0$ and $y = 0$, τ will be maximum

Hence $\boxed{\tau_{max} = \dfrac{4}{3}\dfrac{S}{\pi R^2}}$

$\boxed{\tau_{mean} = \dfrac{S}{\pi R^2}}$ $\Rightarrow$ $\boxed{\tau_{max} = \dfrac{4}{3}\tau_{mean}}$

(3) Symmetrical I-section



Shearing stresses in flanges

$$\boxed{\tau = \frac{S}{IB} \times A\bar{y}}$$

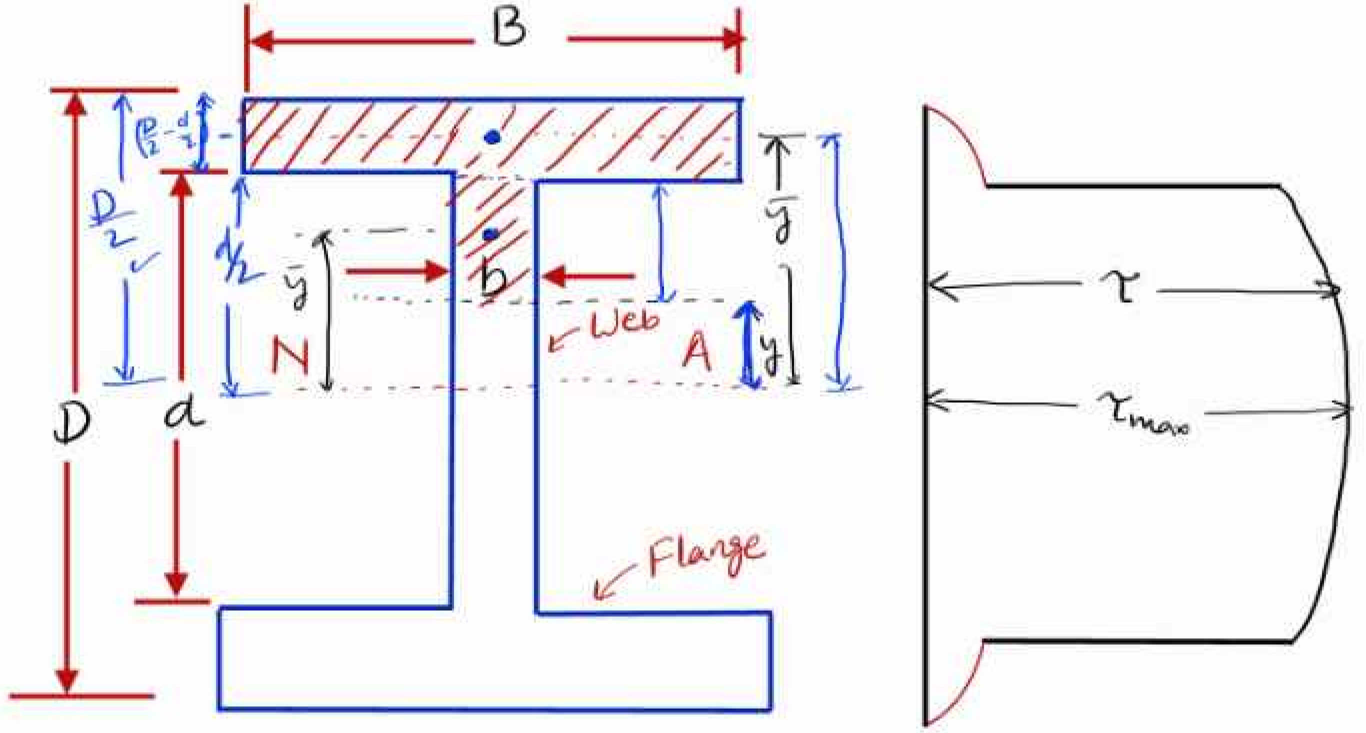
$y + \frac{D}{4} - \frac{y}{2}$

$\frac{y}{2} + \frac{D}{4} = \frac{1}{2}\left(\frac{D}{2} + y\right)$

$$A\bar{y} = B \times \left(\frac{D}{2} - y\right) \times \frac{1}{2}\left(\frac{D}{2} + y\right)$$

$$= \frac{B}{2}\left(\frac{D^2}{4} - y^2\right)$$

$$\tau = \frac{S}{IB} \times \frac{B}{2}\left(\frac{D^2}{4} - y^2\right) = \frac{S}{2I}\left(\frac{D^2}{4} - y^2\right)$$



Shearing stress in web:

$$\tau = \frac{S}{Ib} A\bar{y}$$

$$\frac{D}{2} - \left(\frac{D}{4} - \frac{d}{4}\right) = \left(\frac{D}{4} + \frac{d}{4}\right)$$

$$A\bar{y} = \underbrace{B \cdot \left(\frac{D-d}{2}\right) \times \left(\frac{D+d}{4}\right)}_{\text{flange}} + \underbrace{\left(\frac{d}{2} - y\right) b \times \frac{1}{2}\left(\frac{d}{2} + y\right)}_{\text{Web}}$$

$$y + \frac{\left(\frac{d}{2} - y\right)}{2}$$

$$y + \frac{d}{4} - \frac{y}{2}$$

$$\frac{y}{2} + \frac{d}{2}$$

$$= \frac{1}{2}\left(y + \frac{d}{2}\right)$$

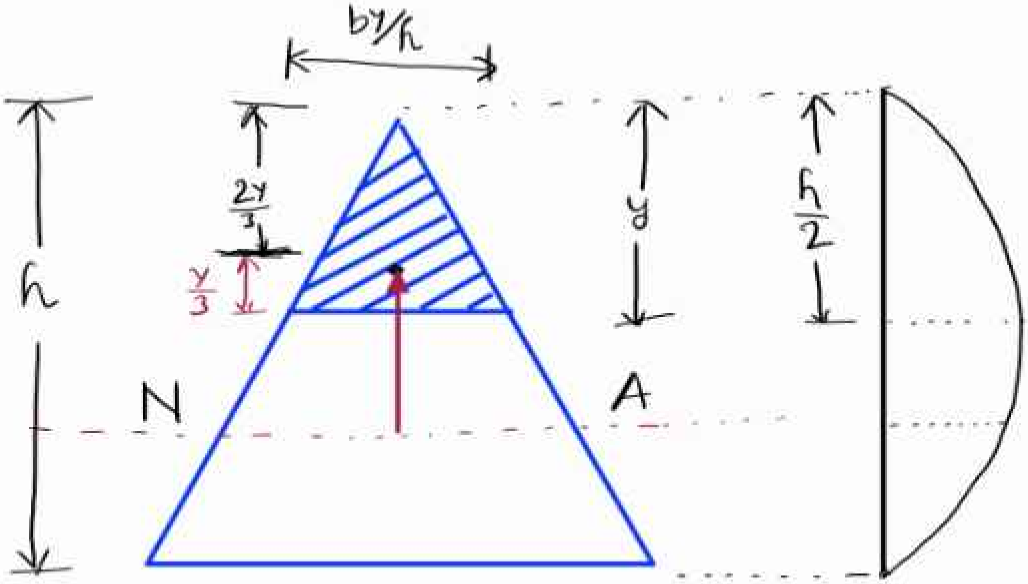
$$= \frac{B}{8}\left(D^2 - d^2\right) + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

$$\boxed{\tau = \frac{S}{8Ib}\left[B\left(D^2 - d^2\right) + b\left(d^2 - 4y^2\right)\right]}$$

The value of τ will be maximum at $y = 0$

$$\boxed{\tau_{max} = \frac{S}{8Ib}\left[B(D^2-d^2) + bd^2\right]}$$

TRIANGULAR SECTION



$$\tau = S \times \frac{A\bar{y}}{Ib} = \frac{S\left(\frac{1}{2}\cdot y \cdot \frac{by}{h}\right)\left(\frac{2}{3}h - \frac{2}{3}y\right)}{\frac{by}{h}\cdot\frac{bh^3}{36}}$$

$$= \frac{12Sy}{bh^3}(h-y) = \frac{12S}{bh^3}(hy-y^2)$$

For maximum value, $\frac{d\tau}{dy} = \frac{d}{dy}(hy-y^2) = 0$

or $h-2y=0$, $y = h/2$

$$\tau_{max} = \frac{12S\,h/2}{bh^3}\left(h - \frac{h}{2}\right) = \frac{3S}{bh} = \frac{3}{2}\cdot\frac{S}{bh/2}$$
$$= 1.5\,\tau_{mean}$$

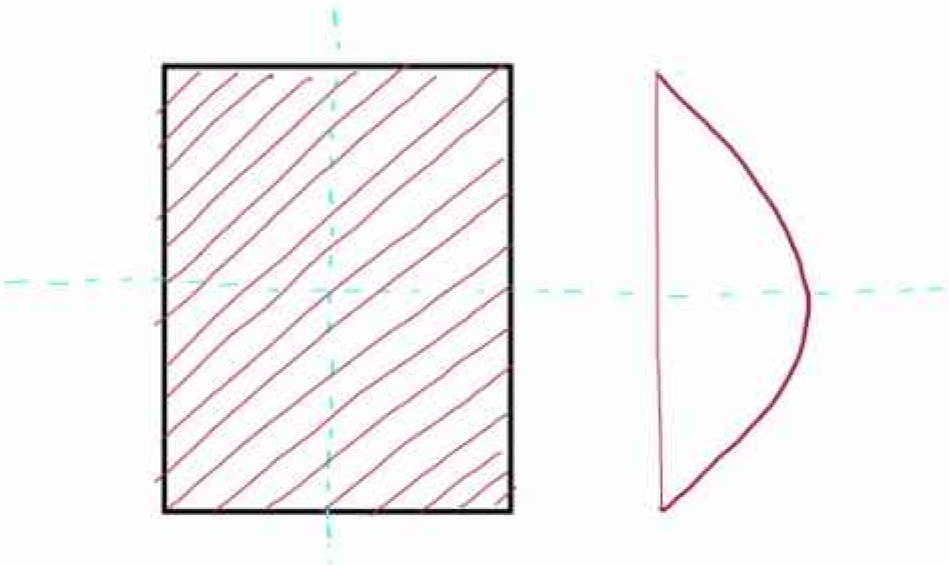
$$\tau_{max} = 1.5\,\tau_{mean}$$

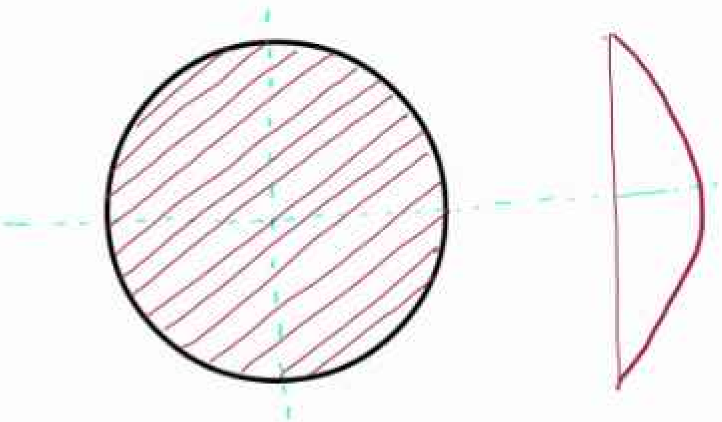
At neutral axis,

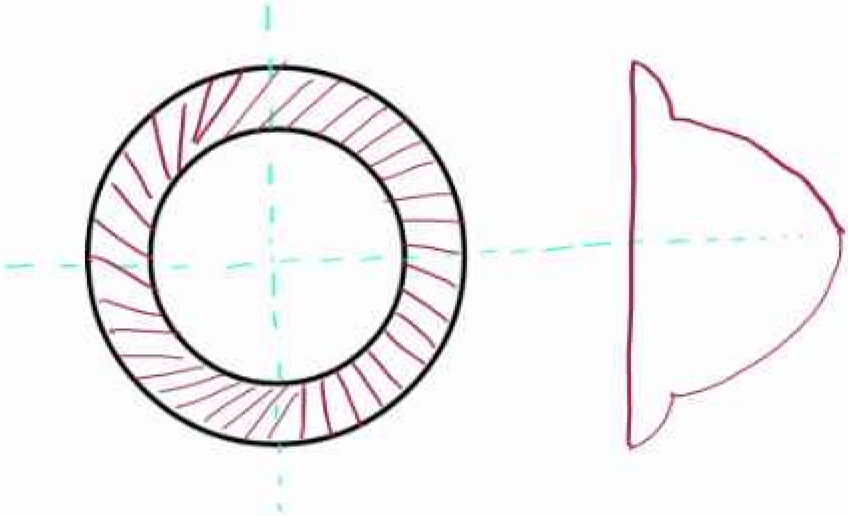
$$\tau = \frac{12S\,(2h/3)}{bh^3}\left(h - \frac{2h}{3}\right)$$

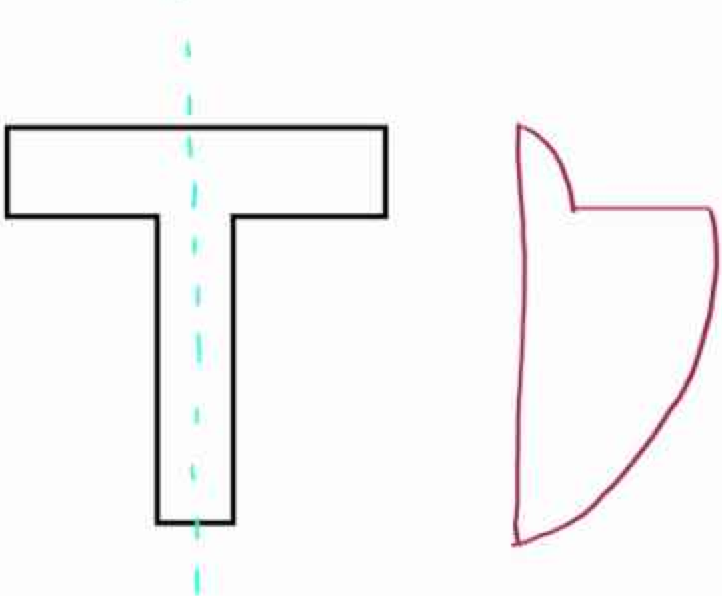
$$\boxed{\tau = \frac{8S}{3bh}}$$

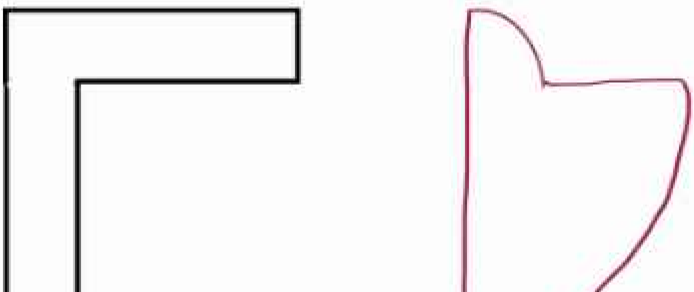
Shear Stress distribution for typical Sections.

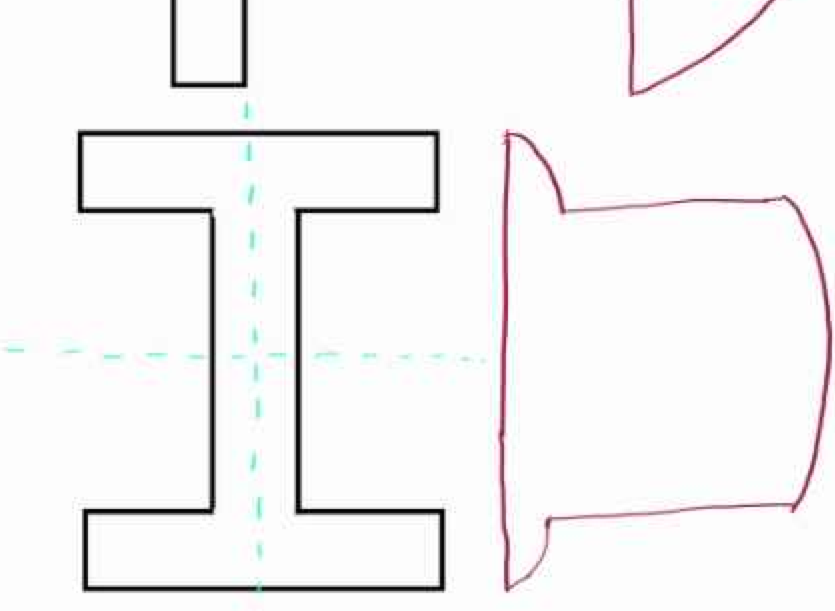


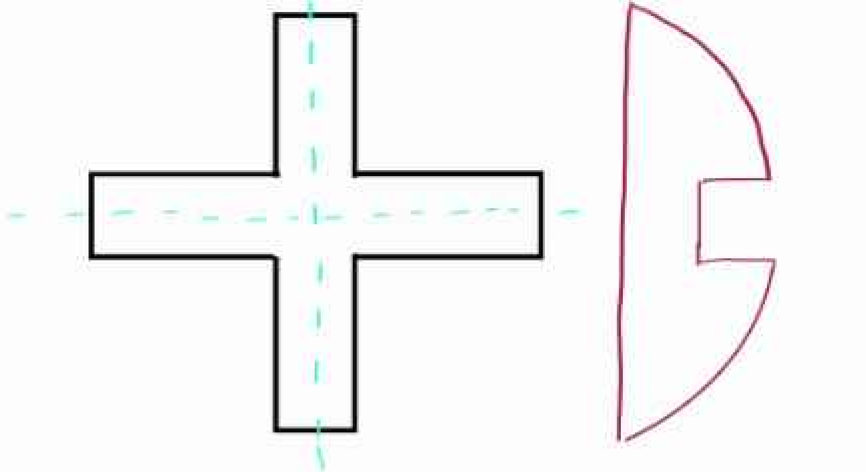


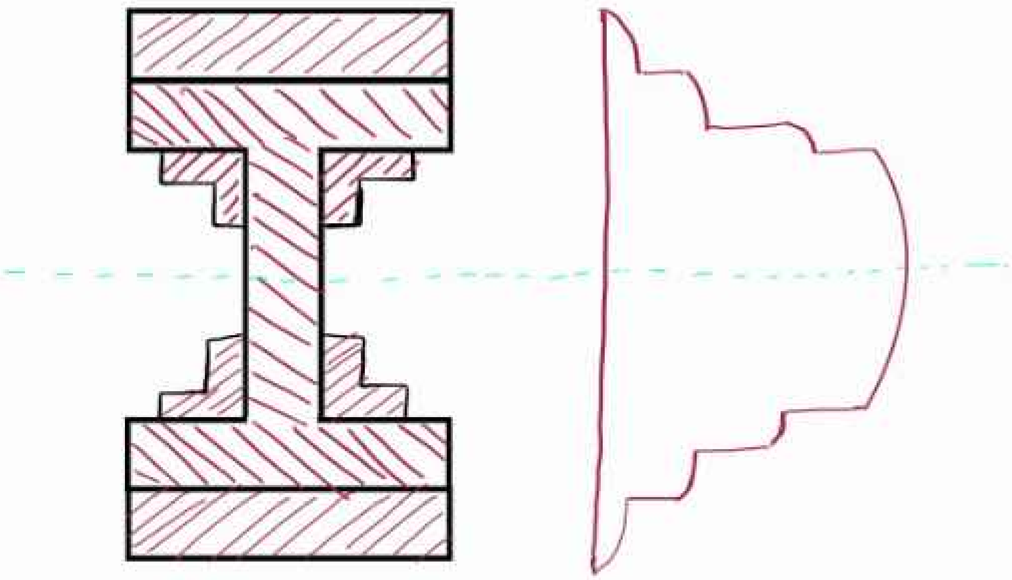








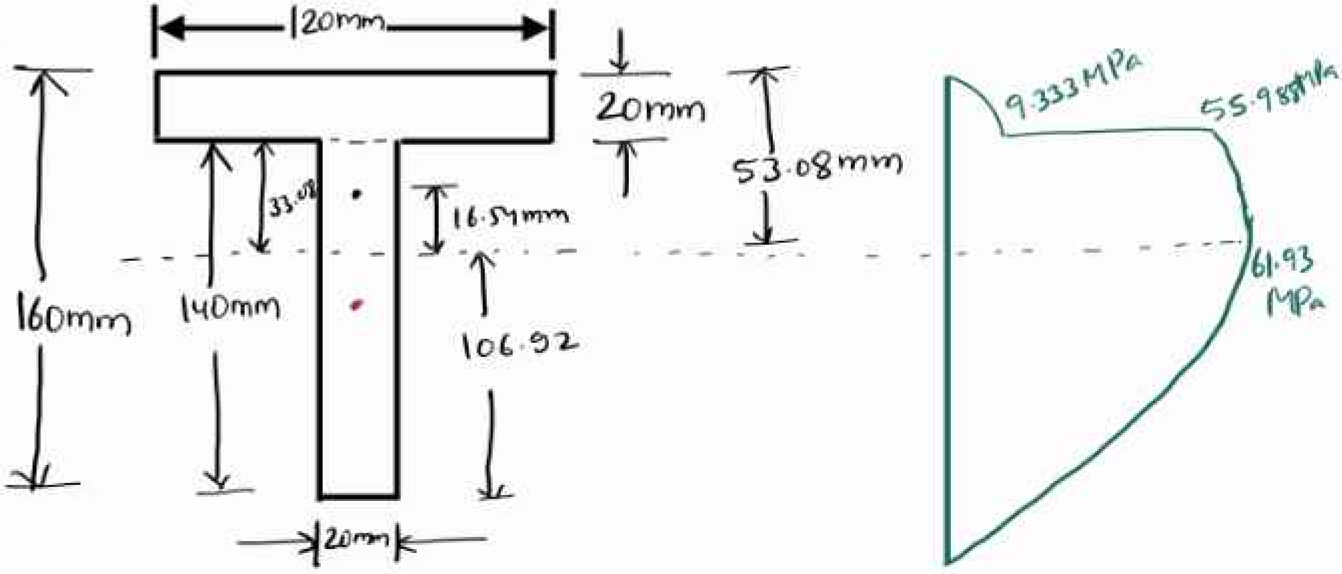


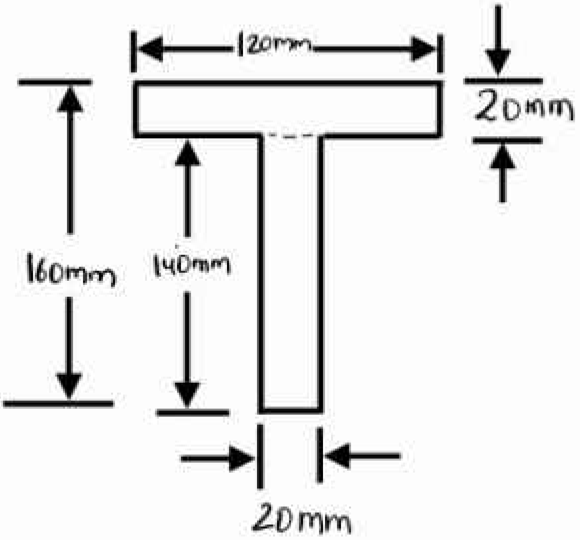


Problem:

A simply supported beam of 2m span carries a uniformly distributed load of 140 KN/m over the whole span. The cross-section of the beam is T-section with a flange width of 120mm, web & flange thickness of 20mm and overall depth of 160mm. Determine the maximum shear stress in the beam and draw the shear stress distribution for the section.

Soln.





Given data

$\omega = 140 \text{ KN/m}$

$L = 2m$

Reaction at each support $= 140 \text{ KN}$

so, maximum shear force $= 140 \text{ KN}$

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2}{A_1 + A_2} = \frac{120 \times 20 \times 150 + 140 \times 20 \times 70}{120 \times 20 + 140 \times 20}$$

$$= \frac{360000 + 198000}{5200}$$

$$\bar{y} = 106.92 \text{ mm}$$

MOI = web + flange

$$= \frac{20\times140^3}{12} + 20\times140\times(106.92-70)^2 + \frac{120\times20^3}{12} + 120\times20\times(53.08-10)^2$$

$$MOI = 12.92\times10^6\ mm^4$$

Shear stress

Shear stress in the flange at the junction of flange and web.

$$\tau = S\frac{A\bar{y}}{Ib} = \frac{140000\times(120\times20)\times(53.08-10)}{12.92\times10^6\times120}$$

$$= 9.333\ MPa$$

Shear stress in the web at the junction

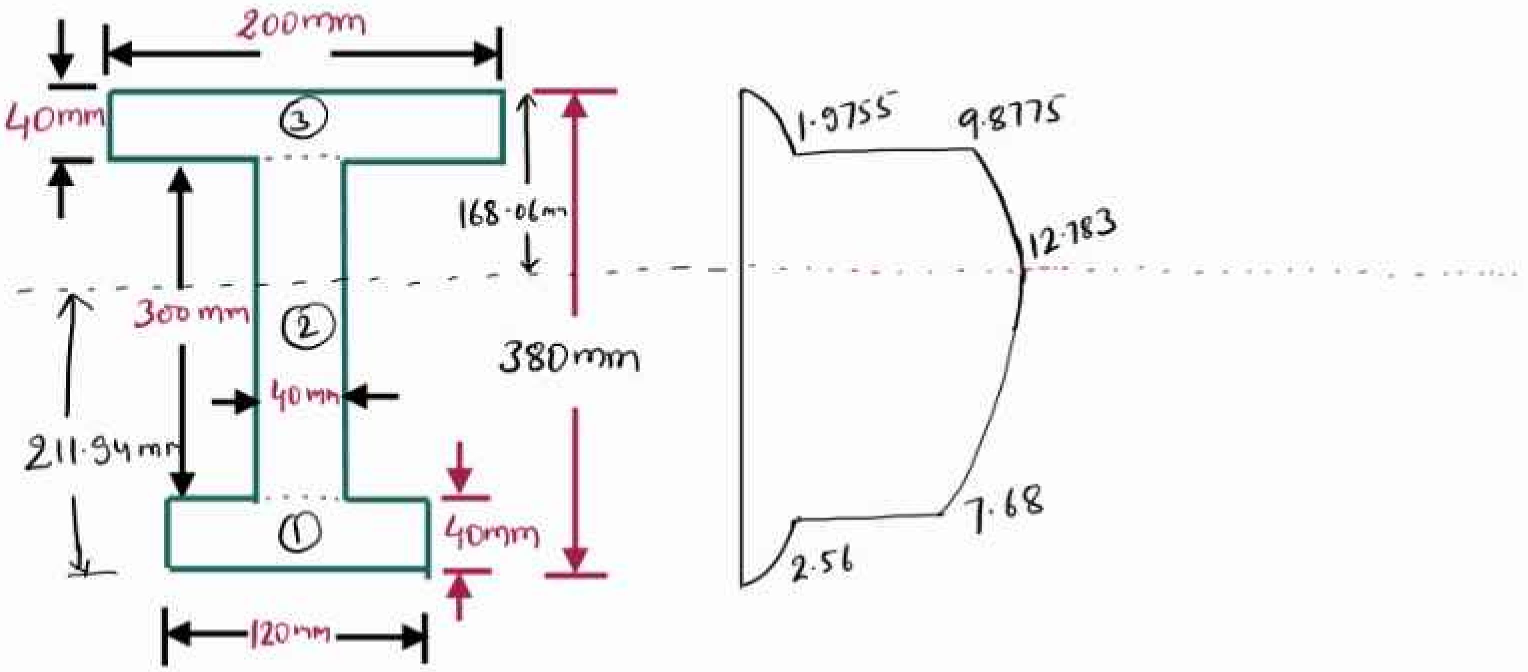
$$= 9.333\times\frac{120}{20} = 55.998\ MPa$$

Maximum shear stress (at neutral axis)

$$= \frac{140000\times[120\times20\times43.08 + 20\times33.08\times16.54]}{12.92\times10^6\times20}$$

$$= 61.93\ MPa$$

Problem: A cast iron-bracket of I-section has its top flange as 200mm x 40mm, bottom flange as 120mm x 40mm and the web as 300mm x 40mm. The overall depth of the section is 380mm. The bracket is subjected to bending. If the maximum stress in the top flange is not exceeded 15MPa, determine the bending moment the section can take. If the beam is subjected to a shear force of 150KN, sketch the shear stress distribution over the depth of the section.



Solution

Given data are,

Shear force, $S = 150$ KN

Shear stress, $\sigma = 15$ MPa

Take reference line as bottom end

$$\text{Centroid}, = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{A_1 + A_2 + A_3}$$

$$= \frac{120 \times 40 \times 20 + 300 \times 40 \times 190 + 200 \times 40 \times 360}{120 \times 40 + 300 \times 40 + 200 \times 40}$$

$$\bar{y} = 211.94 \text{ mm} \quad \text{from bottom}$$

$$\bar{y} = (380 - 211.94) \text{ mm} = 168.06 \text{ mm} \quad \text{from top}$$

$$\text{Moment of Inertia (MOI)} = \frac{120 \times 40^3}{12} + 120 \times 40 \times (211.94 - 20)^2 +$$

$$\frac{300^3 \times 40}{12} + 300 \times 40 \times (211.94 - 190)^2 + \frac{200 \times 40^3}{12} + 200 \times 40 \times (168.06 - 20)^2$$

$$= 177476625.3 + 95776363.2 + 176440775.5$$

$$= 449693764 \text{ mm}^4$$

$$= 449.694 \times 10^6 \text{ mm}^4$$

Bending moment for the upper section

$$\frac{M}{I} = \frac{\sigma_c}{y_c}$$

$$M = \frac{\sigma_c}{y_c} \cdot I = \frac{15 \times 449.694 \times 10^6}{168.06} = 40.13 \text{ KN-m}.$$

$$= 40.13 \text{ kN-m}$$

Bending moment for lower flange.

$$M = \frac{\sigma_t}{y_t} \times I = \frac{18.914 \times 449.694 \times 10^6}{211.93} = 40.13 \text{ KNm}$$

$$= 40.13 \text{ KNm}$$

Shear Stresses

In upper flange at junction with web

$$\tau = \frac{SA\bar{y}}{Ib} = \frac{150 \times 10^3 \times (200 \times 40) \times 148.06}{449.694 \times 10^6 \times 200}$$

$$= 1.9755 \text{ N/mm}^2$$

In web at junction with upper flange

$$\tau = 1.9755 \times \frac{200}{40} = 9.8775 \text{ N/mm}^2$$

At neutral axis,

$$\tau = \frac{SA\bar{y}}{Ib} = \frac{150 \times 10^3 \left[200 \times 40 \times (168.06 - 20) + 40 \times (168.06 - 40) \times \frac{128.06}{2}\right]}{449.694 \times 10^6 \times 40}$$

$$= 12.783 \text{ N/mm}^2$$

In web at junction with lower flange,

$$\tau = \frac{150 \times 10^3 \times (40 \times 120 \times 191.9)}{449.694 \times 10^6 \times 120}$$

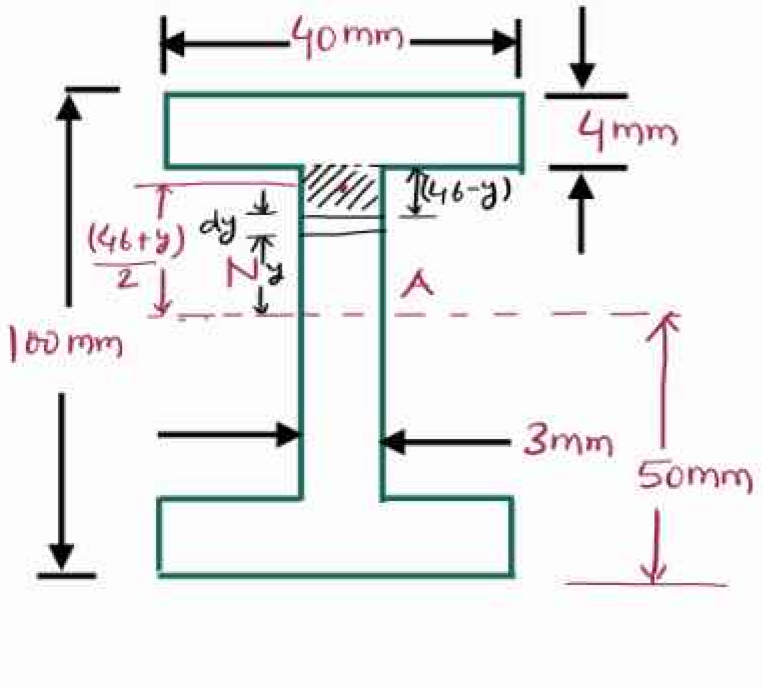
$$\tau = 2.56 \text{ N/mm}^2$$

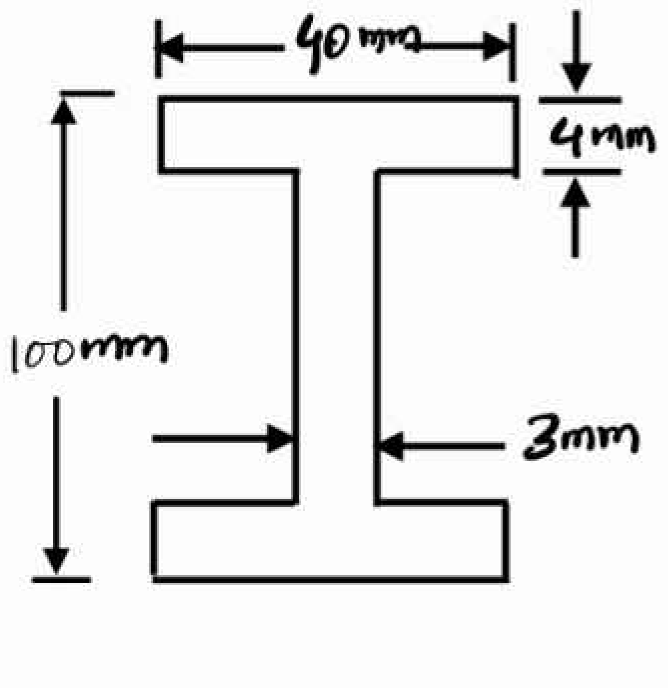
In web at junction with lower flange,

$$\tau = 2.56 \times \frac{120}{40} = 7.68 \text{ N/mm}^2$$

Problem:

A 100mm x 40mm I-beam is subjected to a shear force of 15kN. Find the transverse shear stress at the neutral axis and at the top of the web. Compair it with the mean stress on the assumption of uniform distribution over the web.
What is the percentage of shear force carried by the web?
Moment of Inertia of the section is $1.1 \times 10^6 \text{ mm}^4$, web thickness is 3mm and flange thickness is 4mm.





Shear Stresses

$S = 15\ kN = 15000\ N$

$I = 1.1 \times 10^6\ mm^4$

At neutral axis:

$$\tau = \frac{SA\bar{y}}{Ib} = \frac{15000 \times 40 \times 4 \times 48 + 3 \times 46 \times 23}{3 \times 1.1 \times 10^6}$$

$$= 49.34\ MPa$$

At the top of web

$$\tau = 15000 \times \frac{40 \times 4 \times 48}{3 \times 1.1 \times 10^6} = 34.9\ MPa$$

Shear stress for uniform distribution over web

$$\tau = \frac{150000}{3 \times 92} = 54.35\ MPa$$

Shear force carried by the web

To find the total shear force carried by the web, assume an elementary length of web dy at a distance y from neutral axis.

Shear Stress in elementary length

$$\tau = \frac{SA\bar{y}}{Ib} = 15000 \times \frac{40 \times 4 \times 48 + (46 - y) \times 3 \times [(46 + y)/2]}{3 \times 1.1 \times 10^6}$$

$$= 15000 \times \left[\frac{7680 + 1.5(46^2 - y^2)}{3 \times 1.1 \times 10^6}\right]$$

$$= 15000\left[\frac{7680 + 3174 - 1.5y^2}{3.3\times 10^6}\right]$$

$$\tau = \frac{10854 - 1.5y^2}{220}$$

Shear force in the elementary length = Shear stress × Area

$= \tau \times b \times dy$

Total shear force carried by the web $= \int_{-d/2}^{d/2} \tau\, b\, dy$

$$= \int_{-46}^{46} \frac{(10854 - 1.5y^2)}{220} \times 3 \times dy$$

$$= \frac{1}{73.3}\int_{-46}^{46}(10854 - 1.5y^2)\,dy = \frac{1}{73.3}\left(10854y - \frac{1.5y^3}{3}\right)_{-46}^{46}$$

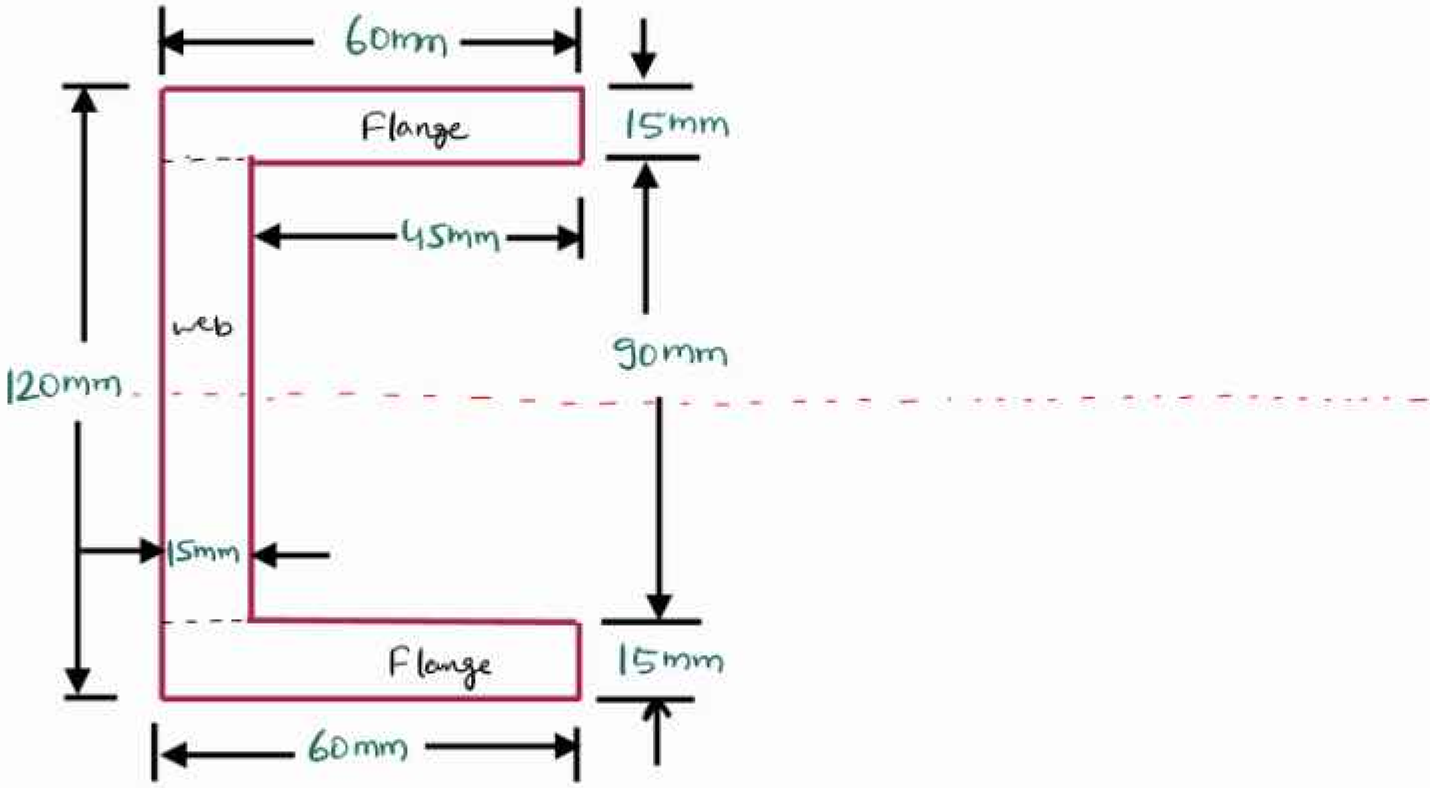
$= 12290\text{ N} = 12.29\text{ KN}$

Percentage shear force carried by the web

$$= 100 - \frac{\text{Total shear force} - \text{shear force carried by web}}{\text{Total shear force}}$$

$$= 100 - \frac{15000 - 12290}{15000} = 82\%.$$

A beam of channel section 120mm x 60mm has uniform thickness of 15mm. Draw shear stress distribution diagram for a vertical section where shearing force is 50kN. Find the ratio between maximum and mean shear stresses.



Solution

Calculation of moment of Inertia

$$I =$$